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Blackbody spectrum from accelerated mirrors with asymptotically inertial trajectories

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Abstract

We reconsider the question of thermal radiation from accelerated mirrors in the model of Fulling and Davies, with the assumption that the mirror moves inertially after a finite acceleration time t_a. We obtain the Planck distribution by calculating the emitted spectrum as $\lim_{t_a \to \infty} N_\omega(t_a)/t_a$, where $N_\omega(t_a)$ is the number of quanta produced in the ω mode on the trajectory with acceleration time t_a.

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1. Introduction

In a recent series of papers [1–3] on the well-known subject of thermal radiation in the model of Fulling and Davies [4, 5], some aspects were brought to attention which, although of fundamental relevance for the issue, seem to have largely remained unremarked in the literature. We can summarize two of the main conclusions established in [1–3] as follows. First, it was shown that, in order to correctly reproduce the blackbody spectrum, it is essential to pay attention to the inertial part of the mirror's trajectory, before it starts to accelerate. Mostly on intuitive grounds, this contribution in the Bogoliubov coefficients was ignored in previous works as being of no relevance for the radiation emitted in the infinite future, with the idea that it corresponds only to a transient phase. As shown in [2, 3], the original calculation of Fulling and Davies for the blackbody spectrum contains two errors, the end result still being the correct one. By a fortunate chance, the missing contribution from the inertial part of the trajectory in their treatment happens to be exactly compensated by a wrong evaluation of the amplitude from the accelerated one.

Secondly, by comparing the Bogoliubov coefficients calculated for trajectories with finite and infinite acceleration times, evidence was shown of their qualitatively distinct behaviour at large frequencies. Based on this, it was concluded that, for a trajectory where the mirror will ultimately revert to a uniform motion, no matter how large the acceleration time and how close to the speed of light the final velocity will be, it is impossible for the two sets of coefficients to assume an identical form [1, 3].

Connected with the two points above, the following picture was suggested in the cited papers. On one hand, we cannot consider, as is commonly done, that it is only the late time part of the trajectory which counts in establishing the Planckian form of the flux, allowing us thus to discern between a 'transient' and a 'relevant' regime at large times in the future. On the other hand, an asymptotically inertial trajectory will necessarily fail to produce this type of spectrum.

Not contradicting the mathematical results in [1–3], we want to show here a way to derive the thermal spectrum which, still, does not make necessary these views. More exactly, we want to present a calculation using trajectories that become *inertial* after a finite acceleration time, with the Planck distribution being obtained in the limit when this time is allowed to approach infinity. Apart from this, we think that our method for calculating the spectrum (which is essentially contained in formula (1)) will provide a good argument in support of the idea that, as traditionally believed, the thermal character can be assigned to the radiation coming from the *late time* part of the accelerated trajectory.

The programme we set before us is as follows. We shall consider a mirror that (i) remains fixed or moves inertially up to the moment $t = 0$, after which (ii) accelerates for a finite time t_a, and then (iii) reverts to a uniform motion, maintaining the same velocity up to the infinite future. Let us recall that a property of the perpetually accelerated trajectories assumed to lead to a Planck spectrum is that they produce a constant flux for times $t \to \infty$. Inspired by this, we shall choose our trajectories so that the emitted flux is constant (i.e. time-independent) for the whole acceleration interval. Our first task will be then to evaluate the total number of particles produced by these motions in the ω mode as a function of the acceleration time t_a. We shall denote this number by $N_\omega(t_a)$.

The next step in our construction requires some discussion. Let us imagine that for a trajectory characterized by a given parameter t_a, a stationary observer analyses the flux as a function of time. According to our own choice above, the energy flux will constantly display, when non-vanishing, the same value. Considering the spectrum of the detected particles, however, there is clearly no reason to expect a similar constancy; the quantum particle with a definite frequency is a non-local object, so most certainly transient effects will show up near the special emission times $t = 0$ and $t = t_a$.

Nevertheless, it is a good question to ask what happens if we let t_a take increasingly large values. Having in mind the time-independence of the flux and, also, the quite unsophisticated nature of our system, it is very plausible to admit that as t_a grows larger and larger the spectrum will also tend to assume some well-defined form, and remain close to it for an increasingly large fraction of the acceleration interval. In particular, in the limit when t_a approaches infinity, this picture naturally allows us to conclude that the vast majority of particles which define the total numbers $N_\omega(t_a)$ will be distributed according to this spectrum.

Thus, it appears a most sensible proposition to identify the emission spectrum from times $t \to \infty$ in the accelerated regime with the following quantity:

$$n_\omega = \lim_{t_a \to \infty} \frac{N_\omega(t_a)}{t_a}. \tag{1}$$

The underlying idea in considering the ratio under the limit is, of course, that due to the constancy of the flux we expect the total number $N_\omega(t_a)$ to diverge linearly with the emission time t_a.

Our second task in the paper will be to evaluate (1), and to see whether we recover thus the spectrum from previous works. As we shall show in the following sections, this proves to be indeed so.

2. Preliminaries

2.1. The trajectories

Let t and x denote the coordinates in two-dimensional Minkowski space, and let us introduce the null coordinates

$$u = t - x \qquad v = t + x. \tag{2}$$

We shall take the accelerated part of the trajectory to be described by the dependence[1]

$$u(v) = -k^{-1}\ln(1 - v/v_H) \tag{3}$$

with k and $v_H > 0$ fixed, and with v in the interval

$$0 < v < v_H. \tag{4}$$

Equation (3) implies that the mirror starts to accelerate in the negative direction from $t = 0$ and $x = 0$, and approaches the speed of light for $v \to v_H$, or

$$t(v) = \tfrac{1}{2}(v - k^{-1}\ln(1 - v/v_H)) \to \infty. \tag{5}$$

The constant v_H fixes the position of the asymptotic part of the trajectory near infinite times, while k determines the value of the local flux (see equation (47); we shall refer throughout the paper to the region at the *right* of the mirror). Depending on the product kv_H the mirror can have different initial velocities at $t = 0$, but this is of no importance for our problem.

To obtain finite acceleration times, we shall introduce an upper limit $v_a < v_H$ for the v coordinate in equation (3), restricting its range to

$$0 < v < v_a. \tag{6}$$

The relation between v_a and the acceleration time t_a can be read from the identity in equation (5). The limit $t_a \to \infty$ will be obviously obtained by letting $v_a \to v_H$.

For v outside the interval (6), we shall assume that the trajectory is uniform. This implies choosing a linear dependence for $u(v)$, which we shall write as

$$u(v) = C_\pm + D_\pm v \tag{7}$$

with $C_\pm$ and $D_\pm$ independent of v. The convention will be that the + case applies for $v > v_a$, and the − case for $v < 0$. The constants in equation (7) are fixed by imposing the natural continuity condition for the mirror's position and velocity at $v = 0$ and $v = v_a$ (which amounts to the continuity of $u(v)$ and its derivative). The exact form of these quantities, however, will play no direct role in our calculation.

2.2. The Bogoliubov coefficients

The first step is to calculate the beta Bogoliubov coefficients connecting the negative and positive frequencies of the *in* and *out* modes

$$\beta(\omega', \omega) = \left(\varphi_{\omega'}^{\text{in}\,*}, \varphi_\omega^{\text{out}}\right) \tag{8}$$

where the parenthesis in the right member denotes the usual scalar product for scalar fields, and where $\varphi_\omega^{\text{in}}$ and $\varphi_\omega^{\text{out}}$ are given by [4, 5]

$$\varphi_\omega^{\text{in}}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}}(\mathrm{e}^{-\mathrm{i}\omega v} - \mathrm{e}^{-\mathrm{i}\omega g(u)}) \tag{9}$$

$$\varphi_\omega^{\text{out}}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}}(\mathrm{e}^{-\mathrm{i}\omega f(v)} - \mathrm{e}^{-\mathrm{i}\omega u}) \tag{10}$$

[1] For times $t \to \infty$ our choice is essentially equivalent to the $x(t) = -\ln\cosh(t)$ trajectory considered in [4, 5].

with $\omega > 0$. We recall that the f function is obtained by taking

$$f(v) = u(v) \tag{11}$$

where $u(v)$ is the trajectory function defined slightly above (g will not appear in our calculation).

To evaluate the scalar product (8), we choose the Cauchy hypersurface to be the null line $u = u_0$, $v_0 < v < \infty$ with u_0 fixed and v_0 such that $u(v_0) = u_0$. We find with this

$$\left(\varphi^{\text{in}\,*}_{\omega'}, \varphi^{\text{out}}_{\omega}\right) = \mathrm{i}\int_{v_0}^{\infty} \mathrm{d}v\, \varphi^{\text{in}}_{\omega'} \overleftrightarrow{\partial_v}\, \varphi^{\text{out}}_{\omega}. \tag{12}$$

The integral (12) can be simplified by eliminating the $\overrightarrow{\partial_v}$ derivative with one integration by parts, from which

$$\left(\varphi^{\text{in}\,*}_{\omega'}, \varphi^{\text{out}}_{\omega}\right) = -2\mathrm{i}\int_{v_0}^{\infty} \mathrm{d}v\, \left(\partial_v \varphi^{\text{in}}_{\omega'}\right)\varphi^{\text{out}}_{\omega} \tag{13}$$

where we have taken into account that no contribution comes from the boundary terms (the modes vanish at the mirror's position and at infinite distances). Introducing equations (9) and (10) in equation (13) and letting $v_0 \to -\infty$, we obtain

$$\beta(\omega', \omega) = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{+\infty} \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}\omega' v - \mathrm{i}\omega f(v)}. \tag{14}$$

(Note that we have ignored the u, v mixing term $\mathrm{e}^{-\mathrm{i}\omega' v - \mathrm{i}\omega u}$ in the integrand; it leads to a quantity proportional to $\delta(\omega')$, thus being of no importance here). Our expression (14) for the beta coefficients is practically the same as that in the calculation of Fulling and Davies, with the only difference being that the integration domain is restricted there to $(0, v_H)$.

Passing to the explicit evaluation of the coefficients, we proceed as follows. We split the v integral as

$$\int_{-\infty}^{+\infty} = \left\{\int_{-\infty}^{0} + \int_{v_a}^{\infty}\right\} + \int_{0}^{v_a} \tag{15}$$

and write, correspondingly,

$$\beta(\omega', \omega) = \beta_u(\omega', \omega) + \beta_a(\omega', \omega) \tag{16}$$

thus separating the contributions from the uniform (β_u) and the accelerated (β_a) parts of the trajectory.

The β_u contribution can be easily calculated, given the linear form for $f(v)$ for v outside the interval $(0, v_a)$. We find[2]

$$\beta_u(\omega', \omega) = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\left(\frac{\mathrm{i}}{\omega' + D_-\omega} - \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega' v_a - \mathrm{i}\omega f(v_a)}}{\omega' + D_+\omega}\right) \tag{17}$$

where we have used the fact that the constants in equation (7) obey, cf the continuity of the trajectory at $v = 0$ and $v = v_a$,

$$C_- = f(0) = 0 \qquad C_+ + D_+ v_a = f(v_a). \tag{18}$$

The β_a contribution is definitely more complicated, so we are content to write for the moment

$$\beta_a(\omega', \omega) = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{0}^{v_a} \mathrm{d}v\, \mathrm{e}^{-\mathrm{i}\omega' v}(1 - v/v_H)^{\mathrm{i}\omega/k}. \tag{19}$$

[2] We introduce a small convergence factor to deal with the $v \to \pm\infty$ limits. Note that the two denominators in the parentheses never vanish, since $\mathrm{d}u/\mathrm{d}v > 0$ implies $D_\pm > 0$.

Our next task will be to extract from equations (17) and (19) an (approximate) expression for the total particle number

$$N_\omega(t_a) = \int_0^\infty \mathrm{d}\omega' |\beta(\omega',\omega)|^2 \tag{20}$$

for large acceleration times $t_a \to \infty$, as a starting point for the evaluation of our limit (1).

3. The asymptotic spectrum

3.1. The divergent $t_a \to \infty$ contribution in $N_\omega(t_a)$

Let us begin by making the observation that if we integrate only the $|\beta_u(\omega',\omega)|^2$ contribution in equation (20), we obtain a logarithmic divergence from frequencies $\omega' \to \infty$. Thus, our first step will be to put $\beta(\omega',\omega)$ in a form that leads, in a visible way, to a finite result.

Let us denote by $I_a(\omega',\omega)$ the integral in equation (19). It is convenient to rewrite it as

$$I_a(\omega',\omega) = v_H\,\mathrm{e}^{-\mathrm{i}\omega' v_H} \int_\varepsilon^1 \mathrm{d}z\,\mathrm{e}^{\mathrm{i}\omega' z} z^{\mathrm{i}\omega/k} \tag{21}$$

which follows from passing to the new variables

$$z = 1 - v/v_H \qquad \varepsilon = 1 - v_a/v_H. \tag{22}$$

Note that the limit $t_a \to \infty$ corresponds now to $\varepsilon \to 0$.

We observe that, if we consider z in equation (21) as a complex variable, the integrand is everywhere analytical for Re $z > 0$. Hence, we can apply the Cauchy theorem for a closed contour in this semiplane. Our choice for the integration path is as follows. We take it to be the rectangle[3] defined by (a) one edge given by the real interval $\varepsilon \leqslant z \leqslant 1$, (b) two edges running parallel with the imaginary axis and extending (1) from ε to $\varepsilon + \mathrm{i}\infty$ and (2) from 1 to $1 + \mathrm{i}\infty$, and (c) the fourth edge closing the contour at $\mathrm{i}\infty$. Noting that no contribution comes from the last edge, we find

$$I_a = I_{b1} - I_{b2} \tag{23}$$

where

$$I_{b1}(\omega',\omega) = \mathrm{i}v_H\,\mathrm{e}^{-\mathrm{i}\omega' v_H(1-\varepsilon)} \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z}(\varepsilon + \mathrm{i}z)^{\mathrm{i}\omega/k} \tag{24}$$

$$I_{b2}(\omega',\omega) = \mathrm{i}v_H \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z}(1 + \mathrm{i}z)^{\mathrm{i}\omega/k}. \tag{25}$$

The next step is to transform the integrals in equations (24) and (25) with one integration by parts using (note that there is no $z \to \infty$ boundary term for the integrands of interest)

$$\int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z} F(z) = \frac{1}{v_H\omega'} F(0) + \frac{1}{v_H\omega'} \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z} F'(z) \tag{26}$$

from which

$$I_{b1}(\omega',\omega) = \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi}}{\omega'} + \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi}}{\omega'}\left(\frac{\mathrm{i}\omega}{k}\right) \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z}(\varepsilon + \mathrm{i}z)^{\mathrm{i}\omega/k-1} \tag{27}$$

$$I_{b2}(\omega',\omega) = \frac{\mathrm{i}}{\omega'} + \frac{\mathrm{i}}{\omega'}\left(\frac{\mathrm{i}\omega}{k}\right) \int_0^\infty \mathrm{d}z\,\mathrm{e}^{-\omega' v_H z}(1 + \mathrm{i}z)^{\mathrm{i}\omega/k-1} \tag{28}$$

[3] This is directly inspired from the calculations of Calogeracos in [2, 3].

where we have introduced

$$e^{i\varphi} = e^{-i\omega' v_H(1-\varepsilon)}\varepsilon^{i\omega/k} = e^{-i\omega' v_a - i\omega f(v_a)}. \tag{29}$$

It is useful to observe for what follows (equation (30)) that the phase $i\,e^{i\varphi}$ is precisely the same as that which appears in the second fraction in the parentheses in equation (17). As a passing remark, it is not hard to realize that this is a direct consequence of the continuity of the trajectory at $v = v_a$, and that a similar observation applies for the identity between the i numerators in equation (28) and that in the first fraction in equation (17), assured by the continuity at $v = 0$.

Now let us make the following grouping in $\beta = \beta_u + \beta_a$. We add the $-$ and $+$ contributions in equation (17) to that corresponding to the first terms in I_{b2} and, respectively, I_{b1}, and we obtain

$$\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\left(\frac{i}{\omega' + D_-\omega} - \frac{i}{\omega'}\right) - \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\left(\frac{i\,e^{i\varphi}}{\omega' + D_+\omega} - \frac{i\,e^{i\varphi}}{\omega'}\right). \tag{30}$$

We can immediately see that equation (30) behaves as $\omega'^{-3/2}$ for $\omega' \to \infty$.

An identical behaviour proves to be valid for the contribution from the remaining terms in I_{b1} and I_{b2}; with a second integration by parts (26) we find that the integrals lead to a second negative power in ω', from which the conclusion follows (see point (1) in the appendix for a detailed check). It is evident that we have obtained thus the well-behaved form of $\beta(\omega', \omega)$ for large ω' frequencies we have looked for.

We turn now to the limit (1). We start with the obvious observation that it is sufficient for our purpose to pay attention only to that part in $|\beta(\omega', \omega)|^2$ which makes $N_\omega(t_a)$ diverge for $t_a \to \infty$. (We might recall here that the total particle number diverges for the forever accelerated trajectories). It is transparent from the quantities in equations (17) and (19) that the divergence can*not* arise in the coefficients themselves, which clearly shows that it is necessary to consider the limit *after* the ω' integration.

A closer look at the ω', ε dependence in equations (27), (28) and (30) shows that the divergence can only arise from the integral term[4] in equation (27). One way to see this is as follows. Let us denote by $\mathcal{B}(\omega', \omega)$ its contribution in $\beta(\omega', \omega)$. We can then prove the following properties (see point (2) in the appendix): (i) the modulus $|\mathcal{B}(\omega', \omega)|$ admits an upper bound of the form $C_1\omega'^{-1/2}$, and (ii) the remaining part up to $\beta(\omega', \omega)$ in modulus, $|\beta(\omega', \omega) - \mathcal{B}(\omega', \omega)|$, also admits an upper bound of the form $C_2\omega'^{-3/2}$, where C_1, C_2 are two quantities independent of $\omega' and\ \varepsilon$. Now let us organize the integrand in equation (20) as

$$N_\omega(t_a) = \int_0^\infty d\omega' |\{\beta(\omega', \omega) - \mathcal{B}(\omega', \omega)\} + \mathcal{B}(\omega', \omega)|^2. \tag{31}$$

Expanding the square in equation (31) and using properties (i) and (ii), it is not difficult to check that the divergence can only come from the pure $\mathcal{B}(\omega', \omega)$ contribution, i.e.

$$\int_0^\infty d\omega' |\mathcal{B}(\omega', \omega)|^2. \tag{32}$$

(All other integrals admit an upper limit independent of ε. It is sufficient to pay attention to the contributions from frequencies $\omega' \to \infty$; see below.)

Thus, we can conclude that for ε approaching zero, or for t_a sufficiently large, the particle number can be approximated to

$$N_\omega(t_a) \sim \frac{(\omega/k^2)}{4\pi^2}\int_0^\infty \frac{d\omega'}{\omega'}\left|\int_0^\infty dz\, e^{-\omega' v_H z}(\varepsilon + iz)^{i\omega/k-1}\right|^2. \tag{33}$$

[4] Note that the integral in equation (28) is actually independent of ε.

Our remaining task is to consider the ratio between equations (33) and (see the identities in equations (5) and (22))

$$t_a \sim -\frac{1}{2k} \ln \varepsilon \tag{34}$$

and find the limit $\varepsilon \to 0$.

3.2. *The limit $\varepsilon \to 0$*

Before proceeding, there are a few comments to be made about formula (33). First, we have to observe that the v_H parameter is of no relevance for the limit. After a change of variable $z \to z/v_H$, we find it reappears only as a multiplicative factor for ε, and (cf the logarithmic dependence in equation (34)) we see that this will have the same effect as that of a finite quantity added to the time t_a.

Secondly, we do not have to worry about what might look like a divergence from frequencies $\omega' \to 0$. The $1/\omega'$ factor must be seen as just an artefact of our approximation. Looking at the expressions in equations (17) and (19), we see immediately that $\beta(\omega', \omega)$ are well behaved in this limit (actually, they vanish). Moreover, we can prove the following property in what concerns the 'low' ω' behaviour of the coefficients; the contributions from frequencies ω' up to *any* fixed value in the particle number (20) will remain *finite* for $\varepsilon \to 0$ (see point (3) in the appendix). Hence, we can justifiably replace the null integration limit $\omega' = 0$ in equation (33) with some non-zero quantity ω'_{inf}, expecting the result for n_ω not to depend on ω'_{inf}.

(It is worth noting at this point that we can see this property as a translation of the common statement in the literature that for the flux from the asymptotic part of the accelerated trajectory (which corresponds here to $\varepsilon \to 0$) 'the high ω' frequencies dominate'. The underlying physical picture is that, because of the increasingly large redshifts due to the reflection on the mirror, we have to count the contributions from zero-point oscillations with increasingly large incoming frequencies ω'.)

Now let us write the limit (1) as

$$n_\omega = -\lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon} \int_{\omega'_{\text{inf}}}^{\infty} \frac{\mathrm{d}\omega'}{\omega'} \mathcal{N}_\omega(\varepsilon\omega') \tag{35}$$

where we have introduced (we set $v_H = 1$ and make $z \to z/\omega'$ in equation (33))

$$\mathcal{N}_\omega(w) = \frac{1}{2\pi^2} \left(\frac{\omega}{k}\right) \left| \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z} (w + \mathrm{i}z)^{\mathrm{i}\omega/k - 1} \right|^2. \tag{36}$$

A convenient method to evaluate the quantity in equation (35) is by observing that, since both the integral and the ln factor diverge for $\varepsilon \to 0$, we can apply the l'Hospital rule. Taking the necessary derivatives with respect to ε, we find (the prime denotes derivation with respect to the argument)

$$n_\omega = -\lim_{\varepsilon \to 0} \varepsilon \int_{\omega'_{\text{inf}}}^{\infty} \mathrm{d}\omega'\, \mathcal{N}'_\omega(\varepsilon\omega') = -\lim_{\varepsilon \to 0} \int_{\varepsilon\omega'_{\text{inf}}}^{\infty} \mathrm{d}w\, \mathcal{N}'_\omega(w). \tag{37}$$

Further observing that $\mathcal{N}_\omega(w)$ vanishes for $w \to \infty$, it is evident from the last integral that equation (35) is equivalent to

$$n_\omega = \lim_{w \to 0} \mathcal{N}_\omega(w). \tag{38}$$

We see at this point that the arbitrary frequency ω'_{inf} disappears, as anticipated, from the spectrum.

Considering now the limit above, we must be aware of the following aspect. Setting directly $w = 0$ in equation (36), as it stands, is problematic due to the z^{-1} behaviour of the integrand near $z = 0$. To sidestep this difficulty, we shall appeal to the following additional analysis. Let us recall that a standard procedure in QFT calculations for assuring well-defined results when dealing with infinitely extended systems, as in our case, is to impose the vanishing condition for the fields at spatial infinity. Let us show that this prescription will lead us to an unambiguous way to evaluate the limit (38).

Clearly, the vanishing condition for a quantum field is equivalent to that for each of its modes, separately. The basic observation to be made in our situation is that, since the ω' frequencies disappeared from equation (36), it is only the ω-labelled modes which must be taken care of. These are the *out* modes (10), and if we further look at the general form of the coefficients (14) we see that, moreover, it is only their v-dependent component which must be considered for our purpose, i.e.

$$e^{-i\omega f(v)}. \tag{39}$$

The next step is to note that, since we deal with the region at the right of the mirror, infinite distances mean $x \to \infty$, and this implies[5] for $f(v)$

$$f(v) \to \infty. \tag{40}$$

The limit (40) tells us that the vanishing of (39) at spatial infinity can be assured by attaching to all frequencies a negative imaginary part, i.e. to make

$$\omega \to \omega - i\lambda \qquad \lambda > 0 \tag{41}$$

in the final formula letting $\lambda \to 0$.

This proves to be crucial for the limit (38); the z^{-1} behaviour in equation (36) now becomes $z^{-1+\lambda/k>-1}$, and in these conditions we can safely set $w = 0$. Thus, it is possible to write

$$\mathcal{N}_\omega(0) = \frac{1}{2\pi^2}\left(\frac{\omega}{k}\right)\left|\int_0^\infty dz\, e^{-z}(iz)^{i\omega/k-1+\lambda/k}\right|^2 \qquad \lambda \to 0. \tag{42}$$

We recognize at this point the Gamma functions in

$$\left|\int_0^\infty dz\, e^{-z}(iz)^{i\omega/k-1+\lambda/k}\right| = e^{-\pi\omega/2k}\,|\Gamma(i\omega/k + \lambda/k)| \tag{43}$$

where there is no problem taking $\lambda = 0$.

Simple further calculations using

$$(i\omega/k)\Gamma(i\omega/k) = \Gamma(i\omega/k + 1) \tag{44}$$

and

$$|\Gamma(i\omega/k + 1)|^2 = \frac{\omega}{k}\frac{\pi}{\sinh \pi\omega/k} \tag{45}$$

finally give

$$n_\omega = \frac{1}{\pi}\frac{1}{e^{2\pi\omega/k} - 1} \tag{46}$$

and this is the result we have sought.

[5] For $v = t + x$ sufficiently large, we reach the uniform motion regime, and here $f(v) = u(v)$ linearly increases with v.

4. Final comments

There is an important observation to be made about the spectrum in equation (46). If we compare it with the standard result in the literature [5], we find that it is larger by a factor of two. This is the moment to mention that the (rightward) emitted energy flux[6] for our trajectories (3) reads

$$\langle T_{uu}(u, v)\rangle_{\text{ren}} = \frac{k^2}{48\pi} \tag{47}$$

and that it is *half* of equation (46) which, integrated over frequencies with ω energy per mode, reproduces this value, i.e.

$$\int_0^\infty \mathrm{d}\omega \left(\frac{n_\omega}{2}\right)\omega = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{\omega}{\mathrm{e}^{2\pi\omega/k}-1} = \frac{k^2}{48\pi}. \tag{48}$$

The origin of this discrepancy is not hard to find. It has to do with the general facts that (1) for a source in motion (the mirror, in our case) the emitted energy per unit of time will not equal that received by a stationary observer, and (2) it is the 'received' energy per time ratio which defines the local flux, identical observations being valid for any other radiated quantity. One can show from simple kinematical considerations that, when the source recedes from the observer with constant velocity v_S and the emission rate is constant in time, the relation between the two ratios is, in evident notations ($\mathcal{Q}$ denotes the radiated quantity, v_S is in units of speed of light and t is still the Minkowski time)

$$\frac{\mathrm{d}\mathcal{Q}^{(\text{rec})}}{\mathrm{d}t} = \frac{1}{1+v_S}\frac{\mathrm{d}\mathcal{Q}^{(\text{em})}}{\mathrm{d}t}. \tag{49}$$

In order to make quantitative our explanation using equation (49), we shall need to invoke the picture regarding the spectrum we have sketched in the introductory section; that is, we shall admit that n_ω can be identified with the number of emitted particles in the ω mode per unit of time at very large times[7] in the accelerated regime, i.e. we can write

$$n_\omega = \frac{\mathrm{d}N_\omega^{(\text{em})}}{\mathrm{d}t} \qquad t \to \infty. \tag{50}$$

Now let us observe that for our trajectories of interest, i.e. with $t_a \to \infty$, the mirror's velocity satisfies $-\mathrm{d}x/\mathrm{d}t \equiv v_S \to 1$ for $t \to \infty$. It is immediate from these that the 'received' spectrum that will be measured by a stationary observer in this limit will be

$$\frac{\mathrm{d}N_\omega^{(\text{rec})}}{\mathrm{d}t} = \frac{n_\omega}{2}. \tag{51}$$

This leads us to the expected consistency with the local field theoretical result for the flux in equation (47).

As a second observation, let us draw attention to a more delicate aspect of the flux, which we have tacitly ignored in our discussion. Let us recall that, if we denote by $\alpha = \alpha(\tau)$ the mirror's proper acceleration, with τ the proper time, the radiated flux can be written

$$\langle T_{uu}(u, v)\rangle_{\text{ren}} = -\frac{1}{12\pi}\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^{-2}\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} \qquad u = u(\tau). \tag{52}$$

It is transparent from equation (52) that at the time $t = t_a$, where α instantaneously jumps from a negative to a null value, a flux with a delta-like profile will be produced. This means

[6] That is, the uu component of the renormalized energy–momentum tensor; all other components of $\langle T_{\mu\nu}\rangle_{\text{ren}}$ identically vanish.

[7] Infinite, in order to reach the supposed steady-state regime $\mathrm{d}N_\omega^{(\text{em})}/\mathrm{d}t = \text{const}$.

that a finite amount of energy will be radiated right at the end of the acceleration regime, which can be easily obtained from equation (52) to be

$$\Delta E = \int_{u_a-0_+}^{u_a+0_+} \mathrm{d}u \langle T_{uu}\rangle_{\mathrm{ren}} = \frac{\alpha}{12\pi}\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^{-1}_{u=u_a-0_+} \qquad u_a = u(v_a). \tag{53}$$

Translating[8] this for our trajectories (3), we find (notably, the result turns out to be independent of v_a)

$$\Delta E = -\frac{k}{24\pi}. \tag{54}$$

The essential point for us is that ΔE remains below a finite upper bound, no matter how large the acceleration interval. This allows us to conclude that only a finite number of particles per mode can be excited during this final emission burst, which means that this number will be of no relevance in the total quantity $N_\omega(t_a) \to \infty$ for $t_a \to \infty$. This assures us that the significance of n_ω is precisely the same as that assumed from the very beginning.

Finally, we would like to point out a possible generalization of our investigation. It arises as a natural question to see how the spectrum will look if we accelerate a semitransparent mirror instead of a perfectly reflecting one. A moving mirror model with this feature, which very likely will allow a similar calculation, was recently constructed in [6], as a direct extension of the model of Fulling and Davies to the case when the mirror is treated as a classical delta-like potential in interaction with the quantum field. As a striking difference with respect to the perfect reflectivity case, it is worth mentioning that the model predicts that, for the same accelerated trajectories as those considered here, the flux will completely vanish in the infinite future [7]. This makes it very plausible that in an analogous calculation the particle number $N_\omega(t_a)$ will remain finite in the limit $t_a \to \infty$, which seems to us as a good indication that the thermal spectrum will fail to appear. We hope to address this and other related questions in a forthcoming paper.

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Appendix

We include here the proof for a series of statements made in the text (points (1)–(3) in section 3).

(1) After one integration by parts, the integrals in equations (27) and (28) lead to an expression of the form

$$\int_0^\infty \mathrm{d}z \cdots = \frac{A}{\omega'} + \frac{B}{\omega'}\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\omega' v_H z}(C + \mathrm{i}z)^{\mathrm{i}\omega/k-2} \tag{A.1}$$

where A and B are independent of ω', and C equals ε or 1. It is sufficient for our purpose to check that the second term in the right member decreases no slower than $1/\omega'$ for $\omega' \to \infty$. To this end, let us observe that for $C > 0$ and $z \geqslant 0$ it is true that

$$|(C + \mathrm{i}z)^{\mathrm{i}\omega/k-2}| \leqslant 1/C^2. \tag{A.2}$$

[8] The useful formulae here are $\mathrm{d}u/\mathrm{d}\tau = \sqrt{u'}$ and $\alpha = -[\sqrt{u'}]'/u'$, with $u' = \mathrm{d}u/\mathrm{d}v$.

We can see from here that the modulus of the C-dependent integral can be majorated by $1/(\omega' v_H C^2)$, which suffices for our point.

(2) Property (i). A simple rearrangement in equation (27) gives, paying attention to the prefactor in equation (19),

$$\mathcal{B}(\omega', \omega) = \frac{1}{2\pi\sqrt{\omega}}\left(\sqrt{\omega'} I_{b1}(\omega', \omega) - \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi}}{\sqrt{\omega'}}\right). \tag{A.3}$$

Considering now the integrand in equation (24), let us observe that for $\varepsilon > 0$ and $z \geqslant 0$ the following inequality holds

$$|(\varepsilon + \mathrm{i}z)^{\mathrm{i}\omega/k}| \leqslant 1. \tag{A.4}$$

Using this in equation (24) we can conclude

$$|I_{b1}(\omega', \omega)| \leqslant 1/\omega' \tag{A.5}$$

and the property follows from applying $|z_1 + z_2| \leqslant |z_1| + |z_2|$ in equation (A.3).

Property (ii). We note first that the difference $\beta(\omega', \omega) - \mathcal{B}(\omega', \omega)$ is the sum between (a) the quantity in equation (30), and (b) the contribution from the second term in equation (28). We apply the triangle inequality to these two terms. The property is evident for quantity (a). For contribution (b), we observe that for $z \geqslant 0$

$$|(1 + \mathrm{i}z)^{\mathrm{i}\omega/k-1}| \leqslant 1. \tag{A.6}$$

This shows that the modulus of the integral in (28) can be majorated by $1/(\omega' v_H)$, which leads to the conclusion.

(3) Let us separate the contribution from $0 < \omega' < \omega'_{\mathrm{inf}}$ in $N_\omega(t_a)$ as

$$N_\omega(t_a)_{\mathrm{low}} = \int_0^{\omega'_{\mathrm{inf}}} \mathrm{d}\omega' |\beta(\omega', \omega)|^2. \tag{A.7}$$

A simple algebra shows that, irrespective of the ω' dependence in $\beta = \beta_u + \beta_a$, we can write

$$\frac{1}{2} N_\omega(t_a)_{\mathrm{low}} \leqslant \int_0^{\omega'_{\mathrm{inf}}} \mathrm{d}\omega' |\beta_u(\omega', \omega)|^2 + \int_0^{\omega'_{\mathrm{inf}}} \mathrm{d}\omega' |\beta_a(\omega', \omega)|^2. \tag{A.8}$$

It is sufficient now to check that each of the two integrands admits an upper bound independent of ε, which remains finite on the integration domain. For the first integrand, this is obvious from equation (17). For the second integrand, if we look at the integral in equation (19) we see that the integrand is a pure phase. We can infer from this (using $v_a < v_H$)

$$|\beta_a(\omega', \omega)|^2 < \frac{1}{4\pi^2} \frac{\omega'}{\omega} v_H \tag{A.9}$$

which concludes the proof.

References

[1] Calogeracos A 2002 *J. Phys. A: Math. Gen.* **35** 3415
[2] Calogeracos A 2002 *J. Phys. A: Math. Gen.* **35** 3435
[3] Calogeracos A 2002 *Int. J. Mod. Phys.* A **17** 1018
[4] Fulling S A and Davies P C 1976 *Proc. R. Soc.* A **348** 393
[5] Birrell N D and Davies P C 1982 *Quantum Fields in Curved Spaces* (Cambridge: Cambridge University Press)
[6] Nicolaevici N 2001 *Class. Quantum Grav.* **18** 619
[7] Nicolaevici N 2001 *Class. Quantum Grav.* **18** 2895